

SOME GENERALIZATIONS OF SMALL PRIMARY MODULES

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Abstract

In this research, a new concepts (D--small primary modules and semi - small primary module) has been presnted, which is a generalizations of a previously studied concept (small primary modules) and a comprehensive study of it in terms of features and properties. We have also given some examples and observations about them. Its relationship with previously studied concepts was studied, and the equivalence, under certain conditions, between these concepts and the new concepts was demonstrated.

Keywords: Primary modules, small primary modules, small submodules, semi primary modules

بعض الاعامات للمقاسات الابتدائية الصغيرة
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المستخلص

في هذا البحث تم تقديم مفهومين جديدين (المقاسات الابتدائية الصغيرة من النمط - d والمقاسات الابتدائية شبه الصغيرة) وهما عبارة عن تعميمات لمفهوم تمت دراسته سابقا (المقاسات الابتدائية الصغيرة) ودراسة شاملة لهما من حيث المميزات والخواص. كما قدمنا بعض الامثلة والملاحظات حولهما. وتم دراسة علاقتهما بالمفاهيم المدروسة سابقا، وتم اثبات التكافؤ، في ظل شروط معينة، بين هذه المفاهيم والمفاهيم الجديدة.

الكلمات المفتاحية: المقاسات الابتدائية، المقاسات الابتدائية الصغيرة، المقاسات الجزئية الصغيرة، المقاسات شبه الابتدائية

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Introduction

In 2005, A. J. Abdul- Alkaik in [1] studied the notions of primary modules as a module X is called "a primary if $\sqrt{\text{ann } X} = \sqrt{\text{ann } P}$ for each proper submodule P of X ". This concept have been generalized by many researchers and have studied these generalizations for example: In 2005, A.J. Abdul- Alkaik in [1] studied the notions of quasi-primary modules as a generalization of primary modules. X is a quasi- primary module if $\text{ann } P$ is a primary ideal of G , for each proper submodule P of X , [1]. In 2017, A. J. Abdul- Alkaik in [2] introduced the concept a small primary module as follows; Where we call "a module X small primary submodule if X is a small primary module if

$\sqrt{\text{ann } X} = \sqrt{\text{ann } P}$ for each proper submodule P small in X , Where "a submodule P of X is called small (notationally, $P \ll X$) if $P + W = X$ for all submodules W of X implies $W = X$ ", [3]. As for this research, we present and study a generalization of the concept a d- small primary module as follows; a module Q is a D- small primary if $\sqrt{\text{ann } Q} = \sqrt{\text{ann } Y}$, $\forall$ non-zero submodule Y d-small in Q . Where "a submodule V of Q is called d--small (notationally, $V \ll_d Q$) if $V + H = Q$ where H is a direct summand sub module of Q , then $H = Q$ ", [4] , and "a submodule B of an G -module Q is called a direct summand of Q iff there exists a submodule C of Q such that

$Q = B \oplus C$, [3]. And "an G -module Q is called a d -hollow module if every non-zero submodule of Q is d -small in Q " [5]. And we present and study a generalization of the concept a semi-small primary module as follows; Where we call an G -module D is a semi-small primary if $\text{ann } P$ is a primary ideal of G for each proper submodule P small in D . In this research, we presented the definitions of d -small primary modules and semi-small primary modules. We also discussed some relationships between them and some types of the previously studied modules and gave the conditions of equivalence between them. We also gave and demonstrated some of characteristics and features of these types of modules.

2- D- Small Primary Modules

Definition 2.1 An G -module D is called d -small primary iff $\sqrt{\text{ann } D} = \sqrt{\text{ann } E}$, $\forall 0 \neq E \ll_d D$.

Remarks and Examples 2.2

1/ Since (0) is the only d -small submodule in D , D is the d -small primary module if D is a semisimple G -module.

2/ D is a d -small primary if it is a primary. The opposite is not true; Z_6 is not primary by [1], but it is d -small primary as a Z -module because it is semisimple.

3/ D is a primary if it is a d -hollow (hollow) d -small primary G -module. Where "an G -module Q is called a hollow module if every non-zero submodule of Q is small in Q " [6].

4/ D is a primary if it is an indecomposable G -module d -small primary G -module.

5/ Given that $(\bar{2}) \ll_d Z_4$, but $2Z = \sqrt{2Z} = \sqrt{\text{ann}(\bar{2})} = \sqrt{\text{ann } Z_4} = \sqrt{4Z} = 2Z$, Z_4 as a Z -module is d -small primary.

6/ Z_{12} as a Z -module is not d -small primary since $(\bar{6}) \ll_d Z_{12}$, but $2Z = \sqrt{2Z} = \sqrt{\text{ann}(\bar{6})} \neq \sqrt{\text{ann } Z_{12}} = \sqrt{12Z} = 6Z$.

7/ Since each proper submodule E of Z is d -small by [5] and $\sqrt{\text{ann}(E)} = \sqrt{\text{ann}(Z)} = 0$, the module Z as Z -module is d -small primary module.

8/ D is a d -small primary if D is a small primary module,

Proof: Since every small submodules is d -small submodules and because D is a small primary, so $\sqrt{\text{ann } D} = \sqrt{\text{ann } E}$ for each proper submodule E small in D , by [2]. As a result, for any proper submodule E d -small in D , $\sqrt{\text{ann } D} = \sqrt{\text{ann } E}$. Consequently, it is a d -small primary.

9/ D is a d -small primary if D is a small prime module.

Proof: D is a small primary by [2] since it is a small prime. D is a d -small primary by (2.2,8) as a result. "A G -module D is called small prime if (0) is small prime sub module", [7].

Theorem 2.3 D is a d -small primary iff $\sqrt{\text{ann } D} = \sqrt{\text{ann}(v)}$, $\forall 0 \neq v \in D$ and $(v) \ll_d D$.

Proof: $\Rightarrow$ It is evident.

$\Leftarrow$ Let $c \in \sqrt{\text{ann } E}$ and $0 \neq E \ll_d D$. Consequently, $c^n v = 0$, $\forall v \in E$ and hence $c \in \sqrt{\text{ann}(v)}$. $(v) \ll_d D$ since $E \ll_d D$ and $(v) \subsetneq E$. Consequently, $\sqrt{\text{ann } D} = \sqrt{\text{ann}(v)}$. However, since $\sqrt{\text{ann } E} \subsetneq \sqrt{\text{ann}(v)}$, $\sqrt{\text{ann } E} \subsetneq \sqrt{\text{ann } D}$. Consequently, D is d -small primary since $\sqrt{\text{ann } D} = \sqrt{\text{ann } E}$.

Proposition 2.4 Assume that every non-zero cyclic is a d -small submodule of D , and that D is a d -small primary module. Then D is a torsion-free

$G/\sqrt{\text{ann}D}$ -module and $G/\sqrt{\text{ann}D}$ is an integral domain.

Proof: To demonstrate that $G/\sqrt{\text{ann}D}$ is an integral domain. Assume that $\bar{c}\bar{e} = \bar{0}$ and $\bar{c}, \bar{e} \in G/\sqrt{\text{ann}D}$. $ce \in \sqrt{\text{ann}D}$ since $ce + \sqrt{\text{ann}D} = 0 + \sqrt{\text{ann}D}$. Therefore, by proposition (2.3), $ce \in \sqrt{\text{ann}(v)}, \forall 0 \neq v \in D$. As a result, either $c \in \sqrt{\text{ann}(ev)}$ or $e \in \sqrt{\text{ann}(cv)}$. since $cen(v) = 0$. However, since D is a d-small primary, $\sqrt{\text{ann}(ev)} = \sqrt{\text{ann}D} = \sqrt{\text{ann}(cv)}$. Therefore, $c \in \sqrt{\text{ann}D}$ or $e \in \sqrt{\text{ann}D}$. Consequently, either $\bar{c} = \bar{0}$ or $\bar{e} = \bar{0}$. $G/\sqrt{\text{ann}D}$ is therefore integral domain. Now, to prove D is a torsion-free. Assume that $0 \neq v \in D$ and $\bar{c} \in G/\sqrt{\text{ann}D}$ such that $\bar{c}v = \bar{0}$. Consequently, $cv = 0$ and $v(c + \sqrt{\text{ann}D}) = 0$. Therefore, according to proposition (2.3), $c \in \sqrt{\text{ann}(v)} = \sqrt{\text{ann}D}$. Consequently, $\bar{c} = \bar{0}$. D is a torsion-free $G/\sqrt{\text{ann}D}$ -module as a result.

Corollary 2,5 Assume that D is a d-small primary module and indecomposable. Then is a torsion-free $G/\sqrt{\text{ann}D}$ -module, and $G/\sqrt{\text{ann}D}$ is integral domain.

Corollary 2.6 Assume that D is a d-small primary faithful G -module that is indecomposable (d-hollow). Then, D is a torsion-free G -module and G is an integral domain.

Proposition 2.7 A non-zero submodule is a d-small primary G -module if D is a d-small primary G -module.

Proof: Assume that D has a submodule $E \neq 0$. Assume that $0 \neq W \ll_d E$. Thus, $W \ll_d D$, [6].

Hence $\sqrt{\text{ann}D} = \sqrt{\text{ann}W}$. But $\sqrt{\text{ann}D} \subsetneq \sqrt{\text{ann}E}$, so $\sqrt{\text{ann}W} \subsetneq \sqrt{\text{ann}E}$. Hence $\sqrt{\text{ann}E} = \sqrt{\text{ann}W}$ and therefore E is a d-small primary.

The example that follows demonstrates that the opposite is not true: Z_6 is a d-small primary Z -module by (2.2, 2). On the other hand, Z_{12} is not a d-small primary Z -module. $\text{Ann}(\bar{6}) = 2Z$ Since $(\bar{6}) \ll_d Z_{12}$ but $\sqrt{\text{ann}Z_{12}} = 12Z \neq \sqrt{\text{ann}(\bar{6})} = 2Z$.

Proposition 2.8 If $\text{Rad}_d(D)$ is a non-zero direct summand d-small primary of a module D and $\sqrt{\text{ann}D} = \sqrt{\text{ann} \text{Rad}_d(D)}$, then D is a d-small primary G -module, where $\text{Rad}_d(D) = \sum \{L \leq D \mid L \ll_d D\}$.

Proof: Suppose that $0 \neq d \in D$ and $(d) \ll_d D$. Then $d \in \text{Rad}_d(D)$, [6] and so $(d) \ll_d \text{Rad}_d(D)$. Therefore $\sqrt{\text{ann} \text{Rad}_d(D)} = \sqrt{\text{ann}(d)}$. But $\sqrt{\text{ann}D} = \sqrt{\text{ann} \text{Rad}_d(D)}$, thus $\sqrt{\text{ann}D} = \sqrt{\text{ann}(d)}$ and hence D is a d-small primary.

Recall that, "an G -module D is called a multiplication if for each submodule E of D there is an ideal A of G such that $E = AD$ ", [8].

Proposition 2.9 Assume that D is a multiplication G -module. Where $\text{End}(D) = S$, D is a d-small primary S -module if it is a d-small primary G -module.

Proof: $\exists c \in \sqrt{\text{ann}E}$ and $c \notin \sqrt{\text{ann}D}$ if $0 \neq E \ll_d D$ and $\sqrt{\text{ann}E} \subsetneq \sqrt{\text{ann}D}$ are assumed. Consequently, $c^n D \neq 0$. Define $\theta: D \rightarrow D$ by $\theta(d) = cv, \forall v \in D$. It is evident that $0 \neq \theta$ is well-defined G -homomorphism. Since D is a d-small primary S -module, $\theta \in \text{ann}_S E \subsetneq$

$\sqrt{\text{ann}_S E} = \sqrt{\text{ann}_S D}$ since $\theta(E) = cE = 0$.
Therefore $\theta^n(D) = 0$, so $\theta = 0$ which is a contradiction. D is therefore the d-small primary G-module since $\sqrt{\text{ann} E} = \sqrt{\text{ann} D}$.

"Recall that an G-module D is called a scalar module if $\forall \varphi \in \text{End}(D); \varphi \neq 0, \exists a \in G, a \neq 0$ such that $\varphi(x) = ax \forall x \in D$ ", [9].

Proposition 2.10 Assume that D is a multiplication finitely generated G-module. Where $\text{End}(D) = S, D$ is a d-small primary S – module if D is a d-small primary G – module.

Proof: Consider the d-small S-submodule of D to be $0 \neq E$. Then $0 \neq E$ is a d-small G-submodule of D . Assume that $\exists \theta \in S, \theta \in \sqrt{\text{ann}_S E}$ and $\notin \sqrt{\text{ann}_S D}$. D is a scalar G-module since it is a multiplication finitely generated, [9]. Thus, $\theta(v) = cv, \forall v \in D$. Consequently, $\theta(E) = cE = 0$, meaning that $c \in \text{ann} E \subsetneq \sqrt{\text{ann} E} = \sqrt{\text{ann} D}$. As a result, $\theta^n(D) = 0$, which is a contradiction since $c^n D = 0$. Thus, $\sqrt{\text{ann}_S E} = \sqrt{\text{ann}_S D}$. D is a d-small primary S – module as a result.

3. SEMI-SMALL PRIMARY MODULES

Definition 3.1 An G-module D is called semi-small primary iff $\text{ann} E$ is a primary ideal, $\forall 0 \neq E \ll D$.

Remarks 3.2 Example and

1- W is a semi- small primary if it is a small primary G-module.

Proof: Consider W to be a small primary G-module. W is therefore a semi – small primary by (3,1), since $\text{ann} E$ is a primary ideal, $\forall 0 \neq E \ll G$, [2].

2- W is a small semi-primary if it is a primary G-module.

Proof: W is a small primary G-module [2] if W is a primary G-module. W is hence a small semi-primary. However, this is not always the case. For instance, Z_6 as a Z –module is small semi-primary but not primary.

3- Z_{p^n} is semi-small primary Z-module. Where n belongs to Z_+ and p is prime number It is straightforward by (2.2,2).

4- Every quasi- small prime G-module is semi-small primary,

Proof: Let W is a quasi-small prime G-module. Then $\text{ann} P$ is a prime ideal, $\forall 0 \neq P \ll W$, [10] and thus, $\text{ann} P$ is a primary ideal [1]. Then W is a semi- small primary by (3.1). However, the opposite is not true, for instance : Z_8 as a Z –module is small semi- primary but not quasi – small prime, [10].

5- Every semisimple (simple) G-module is semi-small primary,

Proof: Let W is a semisimple(simple) G-module, so W is a primary. Then W is a semi- small primary by (2.1)

but the conversely is not true ingeneral, for example : Z_4 as a Z –module is semi- small primary but not simple.

6- If W is a semi- small primary G-module, then a not necessary that $\text{ann} W$ is a primary ideal of G , for example : Z_6 as a Z –module is semi- small primary but $\text{ann} Z_6 = 6Z$ is a not primary ideal.

7- If D is an G-module and $\text{ann} W$ is a primary ideal of G , then W is not a semi- small primary

G-module in general. for example : $Z \oplus Z_{10}$ as a Z -module is not semi- small primary since $\text{ann}(0 \oplus Z_{10}) = 10Z$ is not primary. But $\text{ann}(Z \oplus Z_{10}) = 0$ is a primary ideal.

8- If W is a cyclic G -module and $\text{ann } W$ is a primary ideal of G , then W is a semi- small primary G -module.

Proof: Since W is a cyclic G -module and $\text{ann } W$ is a primary, so W is primary, [1]. Then W is a small semiprimary (3.2, 2).

9- If W is a hollow and semi- small primary G -module, then small primary G -module.

Proposition 3.3

W is a semi- small primary G -module iff $\text{ann}(w)$ is a primary ideal of G , $\forall, 0 \neq w \in W$ and $(w) \ll W$.

Proof: $\Rightarrow$ It is evident.

$\Leftarrow$: Let $0 \neq U \ll W$, to demonstrate that $\text{ann } U$ is a primary ideal of G .

Suppose that $u, v \in G$ such that $uv \in \text{ann } U$ and $0 \neq U \ll W$. If $uvU = 0$ and $uvi = 0$ for every $i \in U$, then $uv \in \text{ann}(i)$. But (i) is a submodule of U and $U \ll W$ implies that $(i) \ll W$, [3]. Since $\text{ann}(i)$ is a primary ideal, either $u \in \text{ann}(i)$ or $v^n \in \text{ann}(i)$ for some $n \in \mathbb{Z}_+$. Consequently, for every $i \in U$, either $v^n i = 0$ or $ui = 0$. Therefore, either $v^n U = 0$ or $uU = 0$, which suggests that either $u \in \text{ann } U$ or $v^n \in \text{ann } U$ for some $n \in \mathbb{Z}_+$. Thus, $\text{ann } U$ is a primary ideal of G . W is hence a semi- small primary G -module.

Theorem 3.4

Let W be a module. If W is a semi- small primary, then $\sqrt{\text{ann } H} = \sqrt{\text{ann } iH}$, $\forall (0) \neq iH, i \in G$ and $H \ll W$.

Proof:

Let $0 \neq H \ll W$. $\text{Ann } H \subsetneq \text{ann } iH$ since $iH \subsetneq H$. Consequently, $\sqrt{\text{ann } H} \subsetneq \sqrt{\text{ann } iH}$. Assume that $a \in \sqrt{\text{ann } iH}$. Since $\forall i \in H$ and $a^n iH = 0$, $a^n i \in \text{ann } H$ for some $n \in \mathbb{Z}_+$. Consequently, $\text{ann } H$ is a primary ideal since W is a semi- small primary. Therefore, for some $k \in \mathbb{Z}_+$, either $(a^n)^k \in \text{ann } H$ or $i \in \text{ann } H$. $iH = 0$ which is a contradiction, if $i \in \text{ann } H$, then $iH = 0$ which is a contradiction. Therefore, $a \in \sqrt{\text{ann } H}$ and so $\sqrt{\text{ann } iH} = \sqrt{\text{ann } H}$. Hence $\sqrt{\text{ann } H} = \sqrt{\text{ann } iH}$.

Corollary 3.5

Assume that W is a module, if D is a semi- small primary then $\sqrt{\text{ann } (u)} = \sqrt{\text{ann } (iu)}$, $\forall (0) \neq iu, i \in G$ and $(u) \ll W$.

Proposition 3.6

If W is a semi- small primary G -module, then a non-zero submodule is a semi- small primary G -module.

Proof. Assume that $0 \neq E \subsetneq W$. Assume that $0 \neq H \ll E$. Thus, $H \ll W$, [3]. then $\text{ann } H$ is a primary ideal of G . Thus, E is a semi- small primary.

Proposition 3.7

States that U is a semi- small primary G -module if $J(U)$ is a direct summand semi- small primary of a G -module U , where $J(U)$ is the Jacobson radical.

Proof. Assume that $(u) \ll U$ and $0 \neq u \in U$. Consequently, $u \in J(U)$. Thus, $(u) \ll J(U)$, [3]. Because of this, U is a semi- small primary and $\text{ann}(u)$ is a primary ideal of G .

Theorem 3.8 Assume that $D \cong Q$. Then, D is a semi- small primary iff Q is a semi- small primary.

Proof.

Let D is a semi- small primary. Since $D \cong Q$, so there exists $\varphi: D \rightarrow Q$ be a G -isomorphism. Assume that $0 \neq K \ll Q$. Hence $\varphi^{-1}(K) \ll D$ and $\varphi^{-1}(K) \neq 0$, [3]. Since D is a semi- small primary, so $\text{ann } \varphi^{-1}(K)$ is a primary ideal. But $D \cong Q$, so $\text{ann } \varphi^{-1}(K) = \text{ann } K$. Hence $\text{ann } K$ is a primary ideal and Q is a semi- small primary module.

By similar prove D is a semi- small primary module.

Proposition 3.9 D is a semi- small primary if Q is a semi- small primary and $\varphi: D \rightarrow Q$ is an G -isomorphism.

Proof:

Assume that $0 \neq K \ll D$. Suppose that $r.u \in \text{ann } K$ such that $r, u \in G$. If $ruK = 0$, then $\varphi(ruK) = ru\varphi(K) = 0$ which suggests that $ru \in \text{ann } \varphi(K)$. However, $K \ll D$, so $\varphi(K) \ll Q$, [3]. Because Q is a semi- small primary, $\text{ann } \varphi(K)$ is also a primary, hence for some $n \in \mathbb{Z}_+$, either $u^n \in \text{ann } \varphi(K)$ or $r \in \text{ann } \varphi(K)$. Therefore, if $r\varphi(K) = 0$ or $u^n\varphi(K) = 0$ for some $n \in \mathbb{Z}_+$, then $\varphi(u^n K) = 0$ or $\varphi(rK) = 0$. However, as $\varphi: D \rightarrow Q$ is a G -isomorphism, either $(u^n K) = 0$ or $(rK) = 0$. Therefore, for some $n \in \mathbb{Z}_+$, either $u^n \in \text{ann } K$ or $r \in \text{ann } K$. D is a semi- small primary as a result.

Theorem 3.10

Assume that $\text{ann } D_1 + \text{ann } D_2 = G$

and that $D = D_1 \oplus D_2$ is a G -module. If both D_1 and D_2 are semi- small primary G -modules, then D is a small semi-primary G -module.

Proof:→

Let $0 \neq K \ll D$. $K = K_1 \oplus K_2$ where K_1 and K_2 are submodules of D_1 and D_2 respectively, since $\text{ann } D_1 + \text{ann } D_2 = G$, [8]. But $K \ll D$, so $K_1 \ll D_1$ and $K_2 \ll D_2$, [3]. Now, as D_1 and D_2 are semi- small primary, $\text{ann } K = \text{ann } (K_1 \oplus K_2) = \text{ann } K_1 \cap \text{ann } K_2$. Thus, the primary ideals are $\text{ann } K_1$ and $\text{ann } K_2$. In G , $\text{ann } K$ is a primary ideal. D is a semi- small primary as a result.

⇐ It follows directly by (3. 6).

Theorem 3.11

Assume that D_1 and D_2 are semi- small primary G -modules and that $D = D_1 \oplus D_2$ is a G -module. Assuming that either $\text{ann } K_1 \subsetneq \text{ann } K_2$ or $\text{ann } K_2 \subsetneq \text{ann } K_1$.

$K_1 \ll D_1$ and $K_2 \ll D_2$, then D is a semi- small primary G -module.

Proof: Since $K_1 \ll D_1$ and $K_2 \ll D_2$, so $(K_1 \oplus K_2) \ll D_1 \oplus D_2$, [3]. We must demonstrate that $\text{ann } (K_1 \oplus K_2)$ is a primary ideal in G in order to establish that D is a semi- small primary G -module. Since $\text{ann } (K_1 \oplus K_2) = \text{ann } K_1 \cap \text{ann } K_2$ and either $\text{ann } K_1 \subsetneq \text{ann } K_2$ or $\text{ann } K_2 \subsetneq \text{ann } K_1$, either $\text{ann } (K_1 \oplus K_2) = \text{ann } K_1$ or $\text{ann } (K_1 \oplus K_2) = \text{ann } K_2$. D_1 and D_2 , however, are modest semi- small primary G -modules. Thus, the primary ideals are $\text{ann } K_1$ and $\text{ann } K_2$.

As a result, Therefore D is a semi- small primary G -module and $\text{ann } (K_1 \oplus K_2)$ is a primary ideal in G .

Theorem 3.12

Assume that I be an ideal of G and D be an G -module such that $I \subsetneq \text{ann } S$. Then D is a semi- small primary G -module if and only if D is a semi- small primary G/I -module.

Proof:

If D is a semi- small primary G -module, to prove D is a semi- small primary G/I -module. if $(u + I)(v + I) \in \text{ann}_{\frac{G}{I}} P$, $(u + I), (v + I) \in G/I, U \ll D$, then $(u.v + I)s = 0, \forall s \in U$, and hence $uvs = 0 \forall s \in U$. Because D is a semi- small primary G -module, $u, v \in \text{ann}_G U$, which is a primary ideal, therefore $v \in \text{ann } U$ or $u^n \in \text{ann } U$ for some $n \in \mathbb{Z}_+$. Therefore, if $vU = (v + I)s = 0, \forall s \in U$ or $u^n U = (u^n + I)s = 0$ for any $n \in \mathbb{Z}_+$, then $\text{ann}_{G/I} U$ is a primary ideal in G/I and D is a semi- small primary G/I -module.

If D is a semi- small primary $\frac{G}{I}$ -module, to prove D is a semi- -small primary G -module. Let $u, v \in G, U \ll D$ and $u.v \in \text{ann}_G U$, then so $uvs = 0 \forall s \in U$. But $U \ll D$ as $\frac{G}{I}$ so $(u + I)(v + I)s = 0, \forall s \in U$. Hence $(u + I)(v + I) \in \text{ann}_{\frac{G}{I}} U$ and $\text{ann}_{\frac{G}{I}} U$ is a primary ideal in $\frac{G}{I}$. either $(v + I) \in \text{ann}_{\frac{G}{I}} U$ or $(u^n + I) \in \text{ann}_{\frac{G}{I}} U$ for some $n \in \mathbb{Z}_+$. Therefore, either $vs = 0, \forall s \in U$, or $u^n s = 0$ for some $n \in \mathbb{Z}_+$. Therefore, for some $n \in \mathbb{Z}_+$, either $u^n \in \text{ann } U$ or $v \in \text{ann } U$. Thus, $\text{ann}_G U$ is a primary ideal and D is a semi- small primary G -module.

Conclusion

The most important findings we have achieved in this research, D is a d-small primary if it is a primary and D is a primary if it is a d-hollow (hollow) d-small primary G -module. D is a primary if it is an indcomposable G -module d-small primary G -module. D is a d-small primary iff $\sqrt{\text{ann} D} = \sqrt{\text{ann}(v)}, \forall 0 \neq v \in D$ and $(v) \ll_d D$.

And W is a semi- small primary if it is a small primary G -module. W is a small semi-primary if it

is a primary G -module. And W is a semi- small primary G -module iff $\text{ann}(w)$ is a primary ideal of $G, \forall 0 \neq w \in W$ and $(w) \ll W$.

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