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(ϕ, ϕ)-Derivation Pair on Semi-Rings

Mohammed Khalid Shahoodh [1]

Abstract

Differential theory is one of the concepts has been considered by many authors since almost thirty years ago. Since that time, many articles have been appeared in this way in order to study this concept about different algebraic structures such as rings or semi-rings. In this paper, we studied the mentioned concept on algebraic structure namely semi-ring. Particularly, we introduced the notation of (ϕ, ϕ)-Derivation pair and studied some of its properties, where ϕ is an automorphism from the semi-ring to itself. We showed that any (ϕ, ϕ)-Derivation pair is a Jordan (ϕ, ϕ)-Derivation pair but the convers is not necessary true. Furthermore, we proved that the sum of (ϕ, ϕ)-Derivation pair (resp. Jordan (ϕ, ϕ)-Derivation pair) is (ϕ, ϕ)-Derivation (resp. Jordan (ϕ, ϕ)-Derivation). Moreover, we showed that if Ş admits Jordan (ϕ, ϕ)-Derivation pair then its (ϕ, ϕ)-Derivation pair.

Keywords: Derivation pair, Semi-ring, Jordan derivation, Differential Algebra, (θ_1, θ_2)-Derivation Pair

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زوج الاشتقاقات- (ϕ, ϕ) على اشباه الحلقات

محمد خالد شاحوذ [1]

الخلاصة

تعتبر نظرية الاشتقاقات من المفاهيم التي تطرق اليها الباحثين منذ فترة تزيد على الثلاثون عاما. منذ ذلك الوقت والكثير من الابحاث التي ظهرت في هذا الاتجاه من اجل دراسة هذا المفهوم حول بنى جبرية مختلفة مثل الحلقة وشبه الحلقة. في هذا البحث تم دراسة هذا المفهوم على تركيبة جبرية تسمى شبه حلقة. على وجه التحديد، قدمنا فكرة زوج الاشتقاقات-(ϕ, ϕ) على شبه الحلقة مع دراسة بعض خواصه ، حيث ϕ تشاكلا ذاتيا من شبه الحلقة الى نفسها. اثبتنا ان مجموع زوج الاشتقاقات-(ϕ, ϕ) هو اشتقاق-(ϕ, ϕ)، وكذلك مجموع زوج اشتقاقات جوردان-(ϕ, ϕ) هو اشتقاق جوردان-(ϕ, ϕ). ايضا بعض الامثلة قد تم عرضها لتوضيح انه زوج اشتقاقات-(ϕ, ϕ) يؤدي الى زوج اشتقاقات جوردان-(ϕ, ϕ) لكن العكس ليس بالضرورة صحيح وكذلك اثبتنا انه اذا كانت شبه الحلقة تحقق زوج اشتقاقات جوردان-(ϕ, ϕ) فانه يكون زوج اشتقاقات-(ϕ, ϕ).

الكلمات المفتاحية: زوج اشتقاقات، شبه حلقة، نظرية الاشتقاقات، زوج اشتقاقات- (θ_1, θ_2) ، اشتقاق جوردان

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Introduction

The study of differential algebra had a great interest by many researchers by discussing some of its properties with different algebraic structures. Some studies have investigated the commutativity of several types of rings in view of derivation and generalized derivations. In [1] Shuliang considered the commutativity of prime ring by using the notation of generalized derivation. Particularly, the author presented some differential conditions under which the ring will be commutative. Ashraf and Rehman [2] checked the commutativity of the prime ring with the concept of derivation. They presented some conditions which can satisfy the commutativity of the ring. On the other hand, the notation of derivation pair was first introduced by Zalar [3-4] and many authors followed this way. The author in [5] provided the notation of (θ_1,θ_2)-derivation pair of rings. He proved that any (θ_1,θ_2)-derivation pair is Jordan (θ_1,θ_2)-derivation pair but not conversely. Some other properties regarding this notation have been also discussed. The notation of Jordan derivation pair on semi-rings has been provided by Thiruveni et. al [6]. They studied some properties of this notation. In this paper, the results of [6] have been extended by providing the notation of (ϕ,ϕ)-derivation pair on semi-rings and discussing some of its basic properties. This article is organized as follows. Section two contains some basic definitions that are needed in this study are given. The main results are given in section three. The conclusions of this study have been stated in section four.

Basic Concepts

This section deals with some basic definitions that are needed in this paper. We begin with the following definition.

Definition 2.1 [6] Let Ṩ be a semi-ring and Ħ be a Ṩ-module. The maps $ɗ,ɠ:Ṩ\longrightarrow Ħ$ are said to be derivation pair and denoted by $(ɗ,ɠ)$, if the conditions below are fulfilled.

$$ɗ(m̵n̵m̵)=ɗ(m̵)n̵m̵+m̵ɠ(n̵)m̵+m̵n̵ɗ(m̵) \text{ for any } n̵,m̵\in Ṩ$$

$$ɠ(m̵n̵m̵)=ɠ(m̵)n̵m̵+m̵ɗ(n̵)m̵+m̵n̵ɠ(m̵) \text{ for any } n̵,m̵\in Ṩ.$$

And they said to be Jordan Derivation pair if

$$ɗ(m̵^3)=ɗ(m̵)m̵^2+m̵ɠ(m̵)m̵+m̵^2ɗ(m̵) \text{ for any } m̵\in Ṩ$$

$$ɠ(m̵^3)=ɠ(m̵)m̵^2+m̵ɗ(m̵)m̵+m̵^2ɠ(m̵) \text{ for any } m̵\in Ṩ.$$

Definition 2.2 [7] A triple $(Ṩ,+,\cdot)$ is called semi-ring if $\cdot$ is disturbutive on $+$.

Definition 2.3 [7] A semi-ring Ṩ is called κ-torsion free if $\kappa m̵=0\Longrightarrow m̵=0$ for any $m̵\in Ṩ$ and $\kappa\in N$.

Main Results

The main results are given in this section and we begin with the observations bellow.

DP = Derivation Pair,

JDP =Jordan Derivation Pair,

(ϕ,ϕ)-DP = (ϕ,ϕ)-Derivation pair,

J-(ϕ,ϕ)-DP = Jordan (ϕ,ϕ)-Derivation pair.

Definition 3.1 Let Ṩ be a semi-ring and Ħ be a Ṩ-module and let $\phi:Ṩ\longrightarrow Ṩ$ be an automorphism of Ṩ. The maps $ɗ,ɠ:Ṩ\longrightarrow Ħ$ which can be denoted by $(ɗ,ɠ)$ are called (ϕ,ϕ)-DP if they obeys the conditions below

$$ɗ(m̵n̵m̵)=ɗ(m̵)\phi(n̵m̵)+\phi(m̵)ɠ(n̵)\phi(m̵)+\phi(m̵n̵)ɗ(m̵) \text{ for any } n̵,m̵\in Ṩ$$

$$ɠ(m̵n̵m̵)=ɠ(m̵)\phi(n̵m̵)+\phi(m̵)ɗ(n̵)\phi(m̵)+\phi(m̵n̵)ɠ(m̵) \text{ for any } n̵,m̵\in Ṩ.$$

And they said to be J-(ϕ,ϕ)-DP if

$$ɗ(ᵯ^3) = ɗ(ᵯ)\phi(ᵯ^2) + \phi(ᵯ)ɠ(ᵯ)\phi(ᵯ) + \phi(ᵯ^2)ɗ(ᵯ) \text{ for any } ᵯ \in Ṩ$$

$$ɠ(ᵯ^3) = ɠ(ᵯ)\phi(ᵯ^2) + \phi(ᵯ)ɗ(ᵯ)\phi(ᵯ) + \phi(ᵯ^2)ɠ(ᵯ) \text{ for any } ᵯ \in Ṩ.$$

Example 3.1 Let $Ṩ = \left\{ \begin{pmatrix} 0 & ᵯ & ᵰ \\ 0 & 0 & ᵯ \\ 0 & 0 & 0 \end{pmatrix} | ᵰ, ᵯ \in N \right\}$ be a semi-ring and

$Ħ = \left\{ \begin{pmatrix} 0 & ᵯ & ᵰ \\ 0 & 0 & ᵯ \\ 0 & 0 & 0 \end{pmatrix} | ᵰ, ᵯ \in Z \right\}$ be a Ṩ-module. Define ɗ,ɠ and ϕ as $ɗ\left(\begin{pmatrix} 0 & ᵯ & ᵰ \\ 0 & 0 & ᵯ \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & ᵰ \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $ɠ\left(\begin{pmatrix} 0 & ᵯ & ᵰ \\ 0 & 0 & ᵯ \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & ᵯ \\ 0 & 0 & 0 \end{pmatrix}$ and $\phi\left(\begin{pmatrix} 0 & ᵯ & ᵰ \\ 0 & 0 & ᵯ \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & ᵯ & ᵰ \\ 0 & 0 & ᵯ \\ 0 & 0 & 0 \end{pmatrix}$. Then, ɗ,ɠ is (ϕ,ϕ)-DP and J-(ϕ,ϕ)-DP.

Remark 3.1 Every (ϕ,ϕ)-DP is a J-(ϕ,ϕ)-DP but not conversely.

Example 3.2 Let Ṩ be a 2-torsion free semi-ring and Ħ be a Ṩ-module. Furthermore, let $ɍ \in Ṩ$ for which $\phi(ᵯ)ɍ\phi(ᵯ) = 0$ for any $ᵯ \in Ṩ$. However, $\phi(ᵯ)\phi(ᵰ)ɍ\phi(ᵯ) \neq 0$ for some $ᵯ \neq ᵰ \in Ṩ$. Define the maps $ɗ, ɠ: Ṩ \longrightarrow Ħ$ as $ɗ(ᵯ) = ɍ\phi(ᵯ)$ and $ɠ(ᵯ) = \phi(ᵯ)ɍ$. Then, (ɗ, ɠ) is a J-(ϕ,ϕ)-DP but not (ϕ,ϕ)-DP.

Let $ᵯ, ᵰ, ɍ \in Ṩ$, then by Definition 3.1, $ɗ(ᵯ^3) = ɗ(ᵯ)\phi(ᵯ^2) + \phi(ᵯ)ɠ(ᵯ)\phi(ᵯ) + \phi(ᵯ^2)ɗ(ᵯ)$ for any $ᵯ \in Ṩ$. So that, $ɗ(ᵯ^3) = ɍ\phi(ᵯ^3)$. From the other side, we have $ɍ\phi(ᵯ)\phi(ᵯ^2) + \phi(ᵯ)\phi(ᵯ)ɍ\phi(ᵯ) + \phi(ᵯ^2)ɍ\phi(ᵯ) = ɍ\phi(ᵯ^3)$. Again, by Definition, 3.1, $ɠ(ᵯ^3) = ɠ(ᵯ)\phi(ᵯ^2) + \phi(ᵯ)ɗ(ᵯ)\phi(ᵯ) + \phi(ᵯ^2)ɠ(ᵯ)$ for any $ᵯ \in Ṩ$. Thus, $ɠ(ᵯ^3) = \phi(ᵯ^3)ɍ$. On the other hand, $\phi(ᵯ)ɍ\phi(ᵯ^2) + \phi(ᵯ)ɍ\phi(ᵯ)\phi(ᵯ) + \phi(ᵯ^2)\phi(ᵯ)ɍ = \phi(ᵯ^3)ɍ$. Therefore, (ɗ, ɠ) is a J-(ϕ,ϕ)-DP. Next, to show that (ɗ, ɠ) is not (ϕ,ϕ)-DP. Now, $ɗ(ᵯᵰᵯ) = ɍ\phi(ᵯᵰᵯ)$ and $ɍ\phi(ᵯ)\phi(ᵰᵯ) + \phi(ᵯ)\phi(ᵰ)ɍ\phi(ᵯ) + \phi(ᵯᵰ)ɍ\phi(ᵯ) = ɍ\phi(ᵯᵰᵯ) + \phi(ᵯᵰ)ɍ\phi(ᵯ) + \phi(ᵯᵰ)ɍ\phi(ᵯ) \neq ɗ(ᵯᵰᵯ)$. Also, $ɠ(ᵯᵰᵯ) = \phi(ᵯᵰᵯ)ɍ$ and $\phi(ᵯ)ɍ\phi(ᵰᵯ) + \phi(ᵯ)ɍ\phi(ᵰ)\phi(ᵯ) + \phi(ᵯᵰ)\phi(ᵯ)ɍ = \phi(ᵯ)ɍ\phi(ᵰᵯ) + \phi(ᵯ)ɍ\phi(ᵰᵯ) + \phi(ᵯᵰᵯ)ɍ \neq ɠ(ᵯᵰᵯ)$. Thus, as required.

Theorem 3.1 The sum of (ϕ,ϕ)-DP is a (ϕ,ϕ)-D. Moreover, the sum of J-(ϕ,ϕ)-DP is a J- (ϕ,ϕ)-D.

Proof: Let (ɗ, ɠ) be a (ϕ,ϕ)-DP of a semi-ring Ṩ with Ħ be a Ṩ-module. By Definition 3.1, we have

$$ɗ(ᵯᵰᵯ) = ɗ(ᵯ)\phi(ᵰᵯ) + \phi(ᵯ)ɠ(ᵰ)\phi(ᵯ) + \phi(ᵯᵰ)ɗ(ᵯ) \text{ for any } ᵰ, ᵯ \in Ṩ \quad (1)$$

$$ɠ(ᵯᵰᵯ) = ɠ(ᵯ)\phi(ᵰᵯ) + \phi(ᵯ)ɗ(ᵰ)\phi(ᵯ) + \phi(ᵯᵰ)ɠ(ᵯ) \text{ for any } ᵰ, ᵯ \in Ṩ. \quad (2)$$

combining (1) and (2), we get

$(ɗ+ɠ)(ᵯᵰᵯ) = (ɗ+ɠ)(ᵯ)\phi(ᵰᵯ) + \phi(ᵯ)(ɗ+ɠ)(ᵰ)\phi(ᵯ) + \phi(ᵯᵰ)(ɗ+ɠ)(ᵯ)$ for any $ᵰ, ᵯ \in Ṩ$. Thereby, $(ɗ+ɠ)$ is (ϕ,ϕ)-D. Also, from Definition 3.1, we have

$$ɗ(ᵯ^3) = ɗ(ᵯ)\phi(ᵯ^2) + \phi(ᵯ)ɠ(ᵯ)\phi(ᵯ) + \phi(ᵯ^2)ɗ(ᵯ) \text{ for any } ᵯ \in Ṩ \quad (3)$$

$$ɠ(ᵯ^3) = ɠ(ᵯ)\phi(ᵯ^2) + \phi(ᵯ)ɗ(ᵯ)\phi(ᵯ) + \phi(ᵯ^2)ɠ(ᵯ) \text{ for any } ᵯ \in Ṩ. \quad (4)$$

Adding (3) with (4), we obtained

$(ɗ+ɠ)(ᵯ^3) = (ɗ+ɠ)(ᵯ)\phi(ᵯ^2) + \phi(ᵯ)(ɗ+ɠ)(ᵯ)\phi(ᵯ) + \phi(ᵯ^2)(ɗ+ɠ)(ᵯ)$ for any $ᵯ \in Ṩ$. Therefore, $(ɗ+ɠ)$ is a J-(ϕ,ϕ)-D. ■

Theorem 3.2 Let $Ṩ$ be a 3-torsion free semi-ring with an identity and Ħ be a $Ṩ$-module. If $Ṩ$ admitting J-(ϕ,ϕ)-DP in which $ɗ(1) = ɠ(1)$, then $ɗ(ᵯ) = ɠ(ᵯ), \forall ᵯ \in Ṩ$.

Proof: From Definition 3.1, we have

$$ɗ(ᵯ^3) = ɗ(ᵯ)\phi(ᵯ^2) + \phi(ᵯ)ɠ(ᵯ)\phi(ᵯ) + \phi(ᵯ^2)ɗ(ᵯ) \text{ for any } ᵯ \in Ṩ \quad (5)$$

$$ɠ(ᵯ^3) = ɠ(ᵯ)\phi(ᵯ^2) + \phi(ᵯ)ɗ(ᵯ)\phi(ᵯ) + \phi(ᵯ^2)ɠ(ᵯ) \text{ for any } ᵯ \in Ṩ \quad (6)$$

Let $\Psi(ᵯ) = ɗ(ᵯ) - ɠ(ᵯ)$ for any $ᵯ \in Ṩ$. Then, by subtracting (6) from (5) we obtained

$$\Psi(ᵯ^3) = \Psi(ᵯ)\phi(ᵯ^2) - \phi(ᵯ)\Psi(ᵯ)\phi(ᵯ) + \phi(ᵯ^2)\Psi(ᵯ) \text{ for any } ᵯ \in Ṩ \quad (7)$$

Linearizing (7), we get

$$\Psi(ᵯ^2ᵰ + ᵰᵯ^2 + ᵯᵰ^2 + ᵰ^2ᵯ + ᵯᵰᵯ + ᵰᵯᵰ) = \Psi(ᵯ)\phi(ᵯᵰ) + \Psi(ᵯ)\phi(ᵰᵯ) + \Psi(ᵯ)\phi(ᵰ^2) +$$

$$\Psi(ᵰ)\phi(ᵯᵰ) + \Psi(ᵰ)\phi(ᵰᵯ) + \Psi(ᵰ)\phi(ᵯ^2) - \phi(ᵯ)\Psi(ᵯ)\phi(ᵰ) - \phi(ᵯ)\Psi(ᵰ)\phi(ᵯ) - \phi(ᵰ)\Psi(ᵯ)$$

$$\phi(ᵯ) - \phi(ᵰ)\Psi(ᵰ)\phi(ᵯ) - \phi(ᵯ)\Psi(ᵰ)\phi(ᵰ) - \phi(ᵰ)\Psi(ᵯ)\phi(ᵰ) + \phi(ᵯᵰ)\Psi(ᵯ) + \phi(ᵰᵯ)\Psi(ᵯ)$$

$$+\phi(ᵰ^2)\Psi(ᵯ) + \phi(ᵯᵰ)\Psi(ᵰ) + \phi(ᵰᵯ)\Psi(ᵰ) + \phi(ᵯ^2)\Psi(ᵰ) \text{ for any } ᵯ, ᵰ \in Ṩ \quad (8)$$

Replace $ᵯ$ by $-ᵯ$ in (8), we have

$$\Psi(ᵯ^2ᵰ + ᵰᵯ^2 - ᵯᵰ^2 - ᵰ^2ᵯ + ᵯᵰᵯ - ᵰᵯᵰ) = \Psi(ᵯ)\phi(ᵯᵰ) + \Psi(ᵯ)\phi(ᵰᵯ) - \Psi(ᵯ)\phi(ᵰ^2) -$$

$$\Psi(ᵰ)\phi(ᵯᵰ) - \Psi(ᵰ)\phi(ᵰᵯ) + \Psi(ᵰ)\phi(ᵯ^2) + \phi(ᵯ)\Psi(ᵯ)\phi(ᵰ) + \phi(ᵯ)\Psi(ᵰ)\phi(ᵯ) + \phi(ᵰ)\Psi(ᵯ)$$

$$\phi(ᵯ) - \phi(ᵰ)\Psi(ᵰ)\phi(ᵯ) - \phi(ᵯ)\Psi(ᵰ)\phi(ᵰ) - \phi(ᵰ)\Psi(ᵯ)\phi(ᵰ) + \phi(ᵯᵰ)\Psi(ᵯ) + \phi(ᵰᵯ)\Psi(ᵯ)$$

$$-\phi(ᵰ^2)\Psi(ᵯ) - \phi(ᵯᵰ)\Psi(ᵰ) - \phi(ᵰᵯ)\Psi(ᵰ) + \phi(ᵯ^2)\Psi(ᵰ) \text{ for any } ᵯ, ᵰ \in Ṩ \quad (9)$$

According to (8) and (9), we get

$$\Psi(ᵯ^2ᵰ + ᵰᵯ^2 + ᵯᵰᵯ) = \Psi(ᵯ)\phi(ᵯᵰ) + \Psi(ᵯ)\phi(ᵰᵯ) + \Psi(ᵰ)\phi(ᵯ^2) - \phi(ᵰ)\Psi(ᵰ)\phi(ᵯ) -$$

$$\phi(ᵯ)\Psi(ᵰ)\phi(ᵰ) - \phi(ᵰ)\Psi(ᵯ)\phi(ᵰ) + \phi(ᵯᵰ)\Psi(ᵯ) + \phi(ᵰᵯ)\Psi(ᵯ) + \phi(ᵯ^2)\Psi(ᵰ) \text{ for any } ᵯ, ᵰ \in Ṩ \quad (10)$$

Putting $ꟽ = 1$ in (10), we have

$$3\Psi(ꞥ) = \Psi(ꞥ)\phi(1) - \phi(ꞥ)\Psi(ꞥ)\phi(1) - \phi(1)\Psi(ꞥ)\phi(ꞥ) + \phi(1)\Psi(ꞥ) \text{ for any } ꞥ \in Ṩ \tag{11}$$

Setting $\phi(ꞥ) = \phi(1) = 1$ in (11), we get

$$3\Psi(ꞥ) = 0 \text{ for any } ꞥ \in Ṩ \tag{12}$$

Since Ṩ is a 3-torsin free, then (12) forces $\Psi(ꞥ) = 0$. Therefore, $ɗ(ꟽ) = ʛ(ꟽ), \forall ꟽ \in Ṩ$. ■

Theorem 3.3 Let Ṩ be 3-torsion free semi-ring and Ħ be a Ṩ-module. If $(ɗ, ʛ)$ is a J-(ϕ,ϕ)-DP of Ṩ then $(\alpha ꞥ + ꞥ\alpha) + (\varepsilon ꞥ + ꞥ\varepsilon) = 0$ for any $ꞥ \in Ṩ$ and $\alpha, \varepsilon \in Ṩ$ with $\phi(ꟽ) = 1$, $ɗ(1) = \alpha$ and $ʛ(1) = \varepsilon$.

Proof: Let $(ɗ, ʛ)$ is a J-(ϕ,ϕ)-DP of Ṩ then by Definition 3.1, we have

$$ɗ(ꟽ^3) = ɗ(ꟽ)\phi(ꟽ^2) + \phi(ꟽ)ʛ(ꟽ)\phi(ꟽ) + \phi(ꟽ^2)ɗ(ꟽ) \text{ for any } ꟽ \in Ṩ \tag{13}$$

$$ʛ(ꟽ^3) = ʛ(ꟽ)\phi(ꟽ^2) + \phi(ꟽ)ɗ(ꟽ)\phi(ꟽ) + \phi(ꟽ^2)ʛ(ꟽ) \text{ for any } ꟽ \in Ṩ \tag{14}$$

Linearizing (14), we get

$$ʛ(ꟽ^2ꞥ + ꟽꞥ + ꞥꟽ) = ʛ(ꞥ)\phi(ꟽ^2) + \phi(ꞥ)ɗ(ꟽ)\phi(ꟽ) + \phi(ꞥꟽ)ʛ(ꟽ) + ʛ(ꟽ)\phi(ꞥꟽ) +$$

$$\phi(ꟽꞥ)ʛ(ꟽ) + ʛ(ꟽ)\phi(ꟽꞥ) + \phi(ꟽ)ɗ(ꟽ)\phi(ꞥ) + \phi(ꟽ^2)ʛ(ꞥ) + \phi(ꟽ)ɗ(ꞥ)\phi(ꟽ) \text{ for any } ꟽ, ꞥ \in Ṩ \tag{15}$$

In (15) put $ꟽ = 1$ and using assumption, we obtained

$$3ʛ(ꞥ) = 2(\alpha ꞥ + ꞥ\alpha) + (\varepsilon ꞥ + ꞥ\varepsilon) + ɗ(ꞥ) + 2ʛ(ꞥ) \text{ for any } ꞥ \in Ṩ \tag{16}$$

That is $ʛ(ꞥ) = 2(\alpha ꞥ + ꞥ\alpha) + (\varepsilon ꞥ + ꞥ\varepsilon) + ɗ(ꞥ)$ for any $ꞥ \in Ṩ$ (17)

By linearizing (13) and applied the same steps we get

$$ɗ(ꞥ) = 2(\alpha ꞥ + ꞥ\alpha) + (\varepsilon ꞥ + ꞥ\varepsilon) + ʛ(ꞥ) \text{ for any } ꞥ \in Ṩ \tag{18}$$

combining (17) with (18), we have

$$ʛ(ꞥ) + ɗ(ꞥ) = 3(\alpha ꞥ + ꞥ\alpha) + 3(\varepsilon ꞥ + ꞥ\varepsilon) + ʛ(ꞥ) + ɗ(ꞥ) \text{ for any } ꞥ \in Ṩ \tag{19}$$

By Definition 2.3, we achieved $(\alpha ꞥ + ꞥ\alpha) + (\varepsilon ꞥ + ꞥ\varepsilon) = 0$ for any $ꞥ \in Ṩ$. ■

Theorem 3.4 Let Ṩ be a semi-ring and Ħ be a Ṩ-module. If $(ɗ, ʛ)$ be a J-(ϕ,ϕ)-DP of Ṩ then, $(ɗ, ʛ)$ is a (ϕ,ϕ)-DP.

Proof: Let $(ɗ, ʛ)$ be a J-(ϕ,ϕ)-DP of Ṩ then, by Definition 3.1, we have

$$ɗ(ꟽ^3) = ɗ(ꟽ)\phi(ꟽ^2) + \phi(ꟽ)ʛ(ꟽ)\phi(ꟽ) + \phi(ꟽ^2)ɗ(ꟽ) \text{ for any } ꟽ \in Ṩ \tag{20}$$

$$ʛ(ꟽ^3) = ʛ(ꟽ)\phi(ꟽ^2) + \phi(ꟽ)ɗ(ꟽ)\phi(ꟽ) + \phi(ꟽ^2)ʛ(ꟽ) \text{ for any } ꟽ \in Ṩ \tag{21}$$

linearize (20),we get

$$\text{ɗ}(m^2n + mnm + nm^2) = \text{ɗ}(n)\phi(m^2) + \phi(n)\text{ɠ}(m)\phi(m) + \phi(nm)\text{ɗ}(m) + \text{ɗ}(m)\phi(nm) + \phi(mn)\text{ɗ}(m) + \text{ɗ}(m)\phi(mn) + \phi(m)\text{ɠ}(m)\phi(n) + \phi(m^2)\text{ɗ}(n) + \phi(m)\text{ɠ}(n)\phi(m) \text{ for any } m,n \in \dot{S} \quad (22)$$

Let $n = 1$ in (22), then

$$3\text{ɗ}(m^2) = \text{ɗ}(1)\phi(m^2) + \phi(1)\text{ɠ}(m)\phi(m) + \phi(m)\text{ɗ}(m) + \text{ɗ}(m)\phi(m) + \phi(m)\text{ɗ}(m) + \text{ɗ}(m)\phi(m) + \phi(m)\text{ɠ}(m)\phi(1) + \phi(m^2)\text{ɗ}(1) + \phi(m)\text{ɠ}(1)\phi(m) \text{ for any } m \in \dot{S} \quad (23)$$

From Theorem 3.3 and by letting $\phi(m^2) = m^2$ and $\phi(1) = 1$, we get

$$3\text{ɗ}(m^2) = \alpha m^2 + \text{ɠ}(m)\phi(m) + \phi(m)\text{ɗ}(m) + \text{ɗ}(m)\phi(m) + \phi(m)\text{ɗ}(m) + \text{ɗ}(m)\phi(m) + \phi(m)\text{ɠ}(m) + m^2\alpha + \phi(m)\varepsilon\phi(m) \text{ for any } m \in \dot{S} \quad (24)$$

That is $3\text{ɗ}(m^2) = 2[\text{ɗ}(m)\phi(m) + \phi(m)\text{ɗ}(m)] + [\text{ɠ}(m)\phi(m) + \phi(m)\text{ɠ}(m)] + (\alpha m^2 + m^2\alpha) + \phi(m)\varepsilon\phi(m)$ for any $m \in \dot{S}$ (25)

Without loss of generality, in (25) let $\text{ɗ}(m)\phi(m) + \phi(m)\text{ɗ}(m) = \text{ɠ}(m)\phi(m) + \phi(m)\text{ɠ}(m) + (\alpha m^2 + m^2\alpha)$ then Eq. (25) will be

$$3\text{ɗ}(m^2) = 3[\text{ɠ}(m)\phi(m) + \phi(m)\text{ɠ}(m)] + 3[(\alpha m^2 + m^2\alpha)] + \phi(m)\varepsilon\phi(m) \text{ for any } m \in \dot{S} \quad (26)$$

That is $\text{ɗ}(m^2) = \text{ɠ}(m)\phi(m) + \phi(m)\text{ɠ}(m) + \phi(m)\varepsilon\phi(m)$ where $(\alpha m^2 + m^2\alpha) = 0$ for any $m \in \dot{S}$ (27)

Linearizing (27), we have

$$\text{ɗ}(mn + nm) = \text{ɗ}(m)\phi(n) + \phi(m)\text{ɗ}(n) + \phi(m)\varepsilon\phi(n) + \text{ɗ}(n)\phi(m) + \phi(n)\text{ɗ}(m) + \phi(n)\varepsilon\phi(m) \text{ for any } m,n \in \dot{S} \quad (28)$$

Replace n by $mn + nm$ in (28), we obtained

$$\text{ɗ}(m^2n + nm^2) + 2\text{ɗ}(mnm) = \text{ɗ}(m)\phi(mn) + \text{ɗ}(m)\phi(nm) + \phi(m)\text{ɗ}(m)\phi(n) + \phi(m^2)\text{ɗ}(n) + \phi(m)\text{ɗ}(n)\phi(m) + \phi(mn)\text{ɗ}(m) + \text{ɗ}(m)\phi(nm) + \phi(m)\text{ɗ}(n)\phi(m) + \text{ɗ}(n)\phi(m^2) + \phi(n)\text{ɗ}(m)\phi(m) + \phi(mn)\text{ɗ}(m) + \phi(nm)\text{ɗ}(m) + \phi(m)\varepsilon\phi(mn) + \phi(m)\varepsilon\phi(nm) + \phi(mn)\varepsilon\phi(m) + \phi(nm)\varepsilon\phi(m) \text{ for any } m,n \in \dot{S} \quad (29)$$

Replace m by m^2 in (28), we have

$$\text{ɗ}(m^2n + nm^2) = \text{ɗ}(m^2)\phi(n) + \phi(m^2)\text{ɗ}(n) + \phi(m^2)\varepsilon\phi(n) + \text{ɗ}(n)\phi(m^2) + \phi(n)\text{ɗ}(m^2) +$$

$\phi(ᵰ)\varepsilon\phi(ᵯ^2)$ for any $ᵯ, ᵰ \in Ṩ$ (30)

From (29) and (30), we obtained

$ɗ(ᵯᵰᵯ) = ɗ(ᵯ)\phi(ᵰᵯ) + \phi(ᵯ)ɗ(ᵰ)\phi(ᵯ) + \phi(ᵯᵰ)ɗ(ᵯ) + \phi(ᵯ)(\varepsilon\phi(ᵰ) + \phi(ᵰ)\varepsilon)\phi(ᵯ) +$

$\phi(ᵯ)(\alpha\phi(ᵰ) + \phi(ᵰ)\alpha)\phi(ᵯ)$ for any $ᵯ, ᵰ \in Ṩ$ (31)

By letting $\phi(ᵰ) = ᵰ$ in (31), we get

$ɗ(ᵯᵰᵯ) = ɗ(ᵯ)\phi(ᵰᵯ) + \phi(ᵯ)ɗ(ᵰ)\phi(ᵯ) + \phi(ᵯᵰ)ɗ(ᵯ) + \phi(ᵯ)(\varepsilon ᵰ + ᵰ\varepsilon)\phi(ᵯ) +$

$\phi(ᵯ)(\alpha ᵰ + ᵰ\alpha)\phi(ᵯ)$ for any $ᵯ, ᵰ \in Ṩ$ (32)

By Theorem 3.3, we have

$ɗ(ᵯᵰᵯ) = ɗ(ᵯ)\phi(ᵰᵯ) + \phi(ᵯ)ɗ(ᵰ)\phi(ᵯ) + \phi(ᵯᵰ)ɗ(ᵯ)$ for any $ᵯ, ᵰ \in Ṩ$ (33)

By Theorem 3.2, $ɗ(ᵰ) = ɠ(ᵰ)$ for any $ᵰ \in Ṩ$. Thus,

$ɗ(ᵯᵰᵯ) = ɗ(ᵯ)\phi(ᵰᵯ) + \phi(ᵯ)ɠ(ᵰ)\phi(ᵯ) + \phi(ᵯᵰ)ɗ(ᵯ)$ for any $ᵯ, ᵰ \in Ṩ$ (34)

By the same way we can prove

$ɠ(ᵯᵰᵯ) = ɠ(ᵯ)\phi(ᵰᵯ) + \phi(ᵯ)ɗ(ᵰ)\phi(ᵯ) + \phi(ᵯᵰ)ɠ(ᵯ)$ for any $ᵰ, ᵯ \in Ṩ$

Therefore, $(ɗ, ɠ)$ is a (ϕ, ϕ)-DP. ■

Theorem 3.5 Let Ṩ be a semi-ring and Ħ be a Ṩ-module. Then, $(ɗ, ɠ)$ is a J-(ϕ, ϕ)-DP iff $(ɗ, ɠ)$ is a (ϕ, ϕ)-DP.

Proof: Let $(ɗ, ɠ)$ be a J-(ϕ, ϕ)-DP, then by Definition 3.1, we have

$ɗ(ᵯ^3) = ɗ(ᵯ)\phi(ᵯ^2) + \phi(ᵯ)ɠ(ᵯ)\phi(ᵯ) + \phi(ᵯ^2)ɗ(ᵯ)$ for any $ᵯ \in Ṩ$ (35)

linearizing (35), we have

$ɗ(ᵯ^2ᵰ + ᵰᵯ^2 + ᵯᵰ^2 + ᵰ^2ᵯ + ᵯᵰᵯ + ᵰᵯᵰ) = ɗ(ᵯ)\phi(ᵯᵰ) + ɗ(ᵯ)\phi(ᵰᵯ) + ɗ(ᵯ)\phi(ᵰ^2) +$

$ɗ(ᵰ)\phi(ᵯᵰ) + ɗ(ᵰ)\phi(ᵰᵯ) + ɗ(ᵰ)\phi(ᵯ^2) - \phi(ᵯ)ɗ(ᵯ)\phi(ᵰ) - \phi(ᵯ)ɗ(ᵰ)\phi(ᵯ) - \phi(ᵰ)ɗ(ᵯ)$

$\phi(ᵯ) - \phi(ᵰ)ɗ(ᵰ)\phi(ᵯ) - \phi(ᵯ)ɗ(ᵰ)\phi(ᵰ) - \phi(ᵰ)ɗ(ᵯ)\phi(ᵰ) + \phi(ᵯᵰ)ɗ(ᵯ) + \phi(ᵰᵯ)ɗ(ᵯ) +$

$\phi(ᵰ^2)ɗ(ᵯ) + \phi(ᵯᵰ)ɗ(ᵰ) + \phi(ᵰᵯ)ɗ(ᵰ) + \phi(ᵯ^2)ɗ(ᵰ)$ for any $ᵯ, ᵰ \in Ṩ$ (36)

Replacing $ᵯ$ by $-ᵯ$ in (36), we have

$ɗ(ᵯ^2ᵰ + ᵰᵯ^2 - ᵯᵰ^2 - ᵰ^2ᵯ + ᵯᵰᵯ - ᵰᵯᵰ) = ɗ(ᵯ)\phi(ᵯᵰ) + ɗ(ᵯ)\phi(ᵰᵯ) - ɗ(ᵯ)\phi(ᵰ^2) -$

$$ɗ(ᵰ)\phi(ᵯᵰ) - ɗ(ᵰ)\phi(ᵰᵯ) + ɗ(ᵰ)\phi(ᵯ^2) + \phi(ᵯ)ɗ(ᵯ)\phi(ᵰ) + \phi(ᵯ)ɗ(ᵰ)\phi(ᵯ) + \phi(ᵰ)ɗ(ᵯ)$$

$$\phi(ᵯ) - \phi(ᵰ)ɗ(ᵰ)\phi(ᵯ) - \phi(ᵯ)ɗ(ᵰ)\phi(ᵰ) - \phi(ᵰ)ɗ(ᵯ)\phi(ᵰ) + \phi(ᵯᵰ)ɗ(ᵯ) + \phi(ᵰᵯ)ɗ(ᵯ) -$$

$$\phi(ᵰ^2)ɗ(ᵯ) - \phi(ᵯᵰ)ɗ(ᵰ) - \phi(ᵰᵯ)ɗ(ᵰ) + \phi(ᵯ^2)ɗ(ᵰ) \text{ for any } ᵯ, ᵰ \in Ṩ \quad (37)$$

According to (36) and (37), we obtained

$$ɗ(ᵯ^2ᵰ + ᵰᵯ^2 + ᵯᵰᵯ) = ɗ(ᵯ)\phi(ᵯᵰ) + ɗ(ᵯ)\phi(ᵰᵯ) + ɗ(ᵰ)\phi(ᵯ^2) - \phi(ᵰ)ɗ(ᵰ)\phi(ᵯ) -$$

$$\phi(ᵯ)ɗ(ᵰ)\phi(ᵰ) - \phi(ᵰ)ɗ(ᵯ)\phi(ᵰ) + \phi(ᵯᵰ)ɗ(ᵯ) + \phi(ᵰᵯ)ɗ(ᵯ) + \phi(ᵯ^2)ɗ(ᵰ) \text{ for any } ᵯ, ᵰ \in Ṩ \quad (38)$$

By letting $\phi(ᵯ^2) = ᵯ^2 = 1$ and $ɗ(ᵰ) = 0$ in (38), we get

$$ɗ(ᵯᵰᵯ) = ɗ(ᵯ)\phi(ᵯᵰ) + ɗ(ᵯ)\phi(ᵰᵯ) - \phi(ᵰ)ɗ(ᵯ)\phi(ᵰ) + \phi(ᵯᵰ)ɗ(ᵯ) + \phi(ᵰᵯ)ɗ(ᵯ) \text{ for any } ᵯ, ᵰ \in Ṩ \quad (39)$$

Now, let $\phi(ᵯᵰ) = 0$ in (39), then we arrive

$$ɗ(ᵯᵰᵯ) = ɗ(ᵯ)\phi(ᵰᵯ) - \phi(ᵰ)ɗ(ᵯ)\phi(ᵰ) + \phi(ᵰᵯ)ɗ(ᵯ) \text{ for any } ᵯ, ᵰ \in Ṩ \quad (40)$$

Replacing $\phi(ᵰ)ɗ(ᵯ)\phi(ᵰ)$ by $-\phi(ᵯ)ɠ(ᵰ)\phi(ᵯ)$ and $\phi(ᵰᵯ)ɗ(ᵯ)$ by $\phi(ᵯᵰ)ɗ(ᵯ)$ in (40), we get

$$ɗ(ᵯᵰᵯ) = ɗ(ᵯ)\phi(ᵰᵯ) + \phi(ᵯ)ɠ(ᵰ)\phi(ᵯ) + \phi(ᵯᵰ)ɗ(ᵯ) \text{ for any } ᵯ, ᵰ \in Ṩ. \quad (41)$$

Similarly, we can prove the other relation. Therefore, $(ɗ, ɠ)$ is a (ϕ,ϕ)-DP. Conversely, assume $(ɗ, ɠ)$ is a (ϕ,ϕ)-DP. Then, by Definition 3.1, we have

$$ɗ(ᵯᵰᵯ) = ɗ(ᵯ)\phi(ᵰᵯ) + \phi(ᵯ)ɠ(ᵰ)\phi(ᵯ) + \phi(ᵯᵰ)ɗ(ᵯ) \text{ for any } ᵯ, ᵰ \in Ṩ \quad (42)$$

Replace ᵰ by ᵯ in (42), we have

$$ɗ(ᵯ^3) = ɗ(ᵯ)\phi(ᵯ^2) + \phi(ᵯ)ɠ(ᵯ)\phi(ᵯ) + \phi(ᵯ^2)ɗ(ᵯ) \text{ for any } ᵯ \in Ṩ$$

Therefore, as required. ■

Conclusion

As a result, this paper presents the notation of (ϕ,ϕ)-DP of semi-rings and proves that the sum of (ϕ,ϕ)-DP (resp. J-(ϕ,ϕ)-DP) is a (ϕ,ϕ)-D (resp. J-(ϕ,ϕ)-D). Also, its proved that $(ɗ, ɠ)$ is a J-(ϕ,ϕ)-DP iff $(ɗ, ɠ)$ is a (ϕ,ϕ)-DP.

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